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## Constrained Coherent States

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### Abstract

Coherent states possess a regularized path integral and give a natural relation between classical variables and quantum operators. Recent work by Klauder and Whiting has included extended variables, that can be thought of as gauge fields, into this formalism. In this paper, I consider the next step, and look at the roll of first class constraints.

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# 1 Introduction

Coherent states were first introduced as non-spreading wave packets for quantum oscillators by Schrödinger in the 1920's. Later this system of states was used for many physical applications such as quantum optics, spin waves, superfluidity, solitons, etc. In addition to these applications, coherent states have been used to address more fundamental issues in quantum mechanics [1] [2]. Klauder and others have been developing a well-defined regularized path integral using coherent state representations. This formalism contains a natural relationship between classical variables and their corresponding quantum operators. Quantum mechanics is also placed on a geometrical foundation. Therefore, a preferred set of coordinates is no longer necessary to quantize a classical system. For a good review see [3] - [5].

In a recent paper by Klauder and Whiting [6] this formalism was extended to include additional variables that can be thought of as gauge degrees of freedom. In this paper, I consider first class constraints and their related gauge symmetries.

## 2 Coherent State Path Integral

A generalized coherent state may be defined in the following way [2]. Let  $G$  be a Lie group acting on a Hilbert space. Let  $\{|\psi_g\rangle\}$  be a system of states where  $|\psi_g\rangle = U(g)|\psi_0\rangle$ ,  $g \in G$ .  $|\psi_0\rangle$  is a fixed vector from the Hilbert space (often called the fiducial vector).  $U(g)$  is a unitary representation of the group  $G$  acting on the Hilbert space. Two states are defined to be equivalent if they differ only by a phase factor. So if  $H$  is the isotropy subgroup in  $G$  such for  $h \in H$ ,

$$U(h)|\psi_0\rangle = e^{i\theta(h)} |\psi_0\rangle, \quad (2.1)$$

then it is clear from this that each inequivalent state is labeled by a member of the left coset space  $G/H$ . For convenience, we shall label the points in this space by  $x \in G/H$  and the coherent state vector by  $|x\rangle$ . These states do not in general form an orthonormal basis. However, they do admit a resolution of unity,

$$\mathbb{I} = \int |x\rangle\langle x| d\mu(x), \quad (2.2)$$

where  $d\mu(x)$  is a positive measure. These states form an (over)complete set of states on the Hilbert space. We can represent a vector in our Hilbert space as a function of  $x$  by defining the function to be

$$\psi(x) \equiv \langle x|\psi\rangle. \quad (2.3)$$

From the resolution of unity (2.2), it can be seen that the inner product on this function space is just the normal inner product on  $L^2$

$$\langle \psi | \phi \rangle = \int \overline{\psi(x)} \phi(x) d\mu(x). \quad (2.4)$$

The overlap function  $\mathcal{K}(x'; x) \equiv \langle x' | x \rangle$  is the reproducing kernel on this space.

$$\begin{aligned} \psi(x') &= \int \mathcal{K}(x'; x) \psi(x) d\mu(x) \\ \mathcal{K}(x''; x) &= \int \mathcal{K}(x''; x') \mathcal{K}(x'; x) d\mu(x') \end{aligned} \quad (2.5)$$

These (2.2 - 2.5) are the basic ingredients for a coherent state representation.

Using these basic ingredients, we can construct a path integral (for more details see [6] and [7]). We start with the matrix element of the Hamiltonian evolution  $\langle x'' | \exp -\frac{i}{\hbar} \hat{H}(t'' - t') | x' \rangle$ . By inserting a resolution of unity at each time slice, we can split the time variable into N pieces and write the total evolution in terms of the evolution between the time slices.

$$\begin{aligned} \langle x'' | e^{-\frac{i}{\hbar} \hat{H}(t'' - t')} | x' \rangle &= \int d\mu(x_1) \langle x'' | e^{-\frac{i}{\hbar} \hat{H}(t'' - t' - \varepsilon)} | x_1 \rangle \langle x_1 | e^{-\frac{i\varepsilon}{\hbar} \hat{H}} | x' \rangle \\ &\vdots \\ \langle x'' | e^{-\frac{i}{\hbar} \hat{H}(t'' - t')} | x' \rangle &= \left( \int \dots \int \prod_{n=1}^N d\mu(x_n) \right) \prod_{n=0}^N \langle x_{n+1} | e^{-\frac{i\varepsilon}{\hbar} \hat{H}} | x_n \rangle \\ x'' &= x_{N+1}, \quad x' = x_0, \quad \varepsilon = (t'' - t') / (N + 1) \end{aligned} \quad (2.6)$$

In the limit ( $\varepsilon \rightarrow 0$ ), if the paths are continuous and differentiable, then we can then make the following approximations.

$$\begin{aligned} \langle x_{n+1} | e^{-\frac{i\varepsilon}{\hbar} \hat{H}} | x_n \rangle &\approx \langle x_{n+1} | \mathbf{1} - \frac{i\varepsilon}{\hbar} \hat{H} | x_n \rangle \\ &\approx \langle x_{n+1} | x_n \rangle \left( 1 - \frac{i\varepsilon}{\hbar} \frac{\langle x_{n+1} | \hat{H} | x_n \rangle}{\langle x_{n+1} | x_n \rangle} \right) \\ &\approx \left( 1 - \varepsilon \langle x_{n+1} | \left( \frac{|x_{n+1}\rangle - |x_n\rangle}{\varepsilon} \right) \right) \left( 1 - \frac{i\varepsilon}{\hbar} \frac{\langle x_{n+1} | \hat{H} | x_n \rangle}{\langle x_{n+1} | x_n \rangle} \right) \\ &\approx (1 - \varepsilon \langle x_{n+1} | \frac{d}{dt} | x_{n+1} \rangle) (1 - \frac{i\varepsilon}{\hbar} \langle x_{n+1} | \hat{H} | x_{n+1} \rangle) \end{aligned} \quad (2.7)$$

We define the symbol  $H(x) \equiv \langle x | \hat{H} | x \rangle$ . Then we can re-exponentiate these two terms, keeping terms up to  $\mathcal{O}(\varepsilon)$ , and place them back into the form above (2.6).

$$\langle x'' | e^{-\frac{i}{\hbar} \hat{H}(t''-t')} | x' \rangle = \int \dots \int \prod_{n=1}^N d\mu(x_n) \prod_{n=1}^{N+1} e^{-\frac{i\varepsilon}{\hbar} (i\hbar \langle x_n | \frac{d}{dt} | x_n \rangle + H(x_n))} \quad (2.8)$$

In the continuum limit, we have the following formal expression for the path integral.

$$\langle x'' | e^{\frac{i}{\hbar} H(t''-t')} | x' \rangle = \int Dx \exp \left\{ \frac{i}{\hbar} \int \left( i\hbar \langle x | \frac{d}{dt} | x \rangle - H(x) \right) dt \right\} \quad (2.9)$$

In addition to the symbol  $H(x)$  above, we can also define another symbol for the operator  $\hat{H}$ . This symbol is implicitly defined in terms of the spectral representation of the operator [1].

$$\hat{H} = \int h(x) |x\rangle \langle x| d\mu(x) \quad (2.10)$$

This symbol  $h(x)$  is called the lower symbol while  $H(x)$  is called the upper symbol. If such a representation exists and is well defined (see [8]) then we have another way to derive the path integral. Let us consider the infinitesimal time evolution operator in terms of the lower symbol. With the resolution of unity and the above definition (2.10), we can write this operator in the following form.

$$\mathbb{I} - \frac{i\varepsilon}{\hbar} \hat{H} = \int \left[ 1 - \frac{i\varepsilon}{\hbar} h(x) \right] |x\rangle \langle x| d\mu(x) \quad (2.11)$$

Then by exponentiating both sides, dropping terms of order  $\varepsilon^2$ , and apply repeated operations of this operator, we can build up a finite time displacement operator.

$$e^{-\frac{i}{\hbar} \hat{H}T} = \int \dots \int \prod_{n=1}^N e^{-\frac{i\varepsilon}{\hbar} h(x_n)} |x_n\rangle \langle x_n| d\mu(x_n) \quad \text{where} \quad T = N\varepsilon \quad (2.12)$$

The Hamiltonian evolution matrix then takes the form

$$\langle x'' | e^{-\frac{i}{\hbar} \hat{H}(t''-t')} | x' \rangle = \int \dots \int \prod_{n=1}^{N+1} \langle x_n | x_{n-1} \rangle \prod_{n=1}^N e^{-\frac{i\varepsilon}{\hbar} h(x_n)} d\mu(x_n). \quad (2.13)$$

If we make similar approximations as we did in (2.7), in the continuum limit, we have the same form of the path integral as in (2.9) but with  $h(x)$  replacing  $H(x)$ .

$$\langle x'' | e^{\frac{i}{\hbar} H(t''-t')} | x' \rangle = \int Dx \exp \left\{ \frac{i}{\hbar} \int \left( i\hbar \langle x | \frac{d}{dt} | x \rangle - h(x) \right) dt \right\} \quad (2.14)$$

This second form of the symbol of  $\hat{H}$  is related to the first form by

$$H(x) = \int h(x') |\langle x | x' \rangle|^2 d\mu(x'). \quad (2.15)$$

In general these two symbols will not be equivalent. However for a suitable choice of coherent states, the difference between them will only be of order  $\hbar$ . Therefore, when we take the stationary phase approximation to the path integral, both  $h(x)$  and  $H(x)$  will lead to the same equations of motion.

Although there have been attempts to regularize the ordinary configuration space path integral by introducing additional terms (see [9] for a good review) these attempts have met with limited success. However the coherent state path integral is inherently a path integral over the phase space, and because of this it is possible to regularize with path integral by changing the measure to a pinned Wiener measure. This measure originally came from the study of Brownian motion. The probability density of a particle undergoing Brownian motion is governed by the diffusion equation. The equation for the density  $\rho(t''; t')$  at  $t''$  starting with initial data at time  $t'$  is given by

$$\frac{\partial \rho(t''; t')}{\partial t''} = \frac{1}{2} \nu \Delta \rho(t''; t'), \quad (2.16)$$

where  $\Delta$  is the Laplace-Beltrami operator. For an example, let the metric be a flat metric ( $d\sigma^2 = dp^2 + dq^2$ ), then the fundamental solution of this equation is given by

$$\rho(t''; t') = \frac{1}{2\pi\nu(t'' - t')} \exp \left[ -\frac{(p'' - p')^2 + (q'' - q')^2}{2\nu(t'' - t')} \right]. \quad (2.17)$$

The most important property of this solution for the Wiener measure is that the density  $\rho(t''; t')$  posses the following product rule [8].

$$\rho(t'''; t') = \int dp'' dq'' \rho(t'''; t'') \rho(t''; t') \quad (2.18)$$

This product rule can be repeated to form a lattice in the time direction.

$$\begin{aligned} \rho(t''; t') &= \int \left( \prod_{i=1}^N dp_i dq_i \right) \left( \frac{1}{2\pi\nu\varepsilon} \right)^N \left( \exp \sum_{i=0}^N -\frac{(p_{i+1} - p_i)^2 + (q_{i+1} - q_i)^2}{2\nu\varepsilon} \right) \\ (q'', p'') &= (q_{N+1}, p_{N+1}), \quad (q', p') = (q_0, p_0), \quad \varepsilon = (t'' - t')/N \end{aligned} \quad (2.19)$$

In the continuum limit, we now have a formal expression for the Wiener measure which is pinned for both  $q$  and  $p$  at  $t''$  and  $t'$ .

$$d\mu_W^\nu(p, q) = \mathcal{N} e^{-\frac{1}{2\nu} \int \dot{p}^2 + \dot{q}^2 dt} Dq Dp \quad (2.20)$$

Writing this in a more general way to include other choices for the metric, we have

$$d\mu_W^\nu(x) = \mathcal{N} e^{-\frac{1}{2\nu} \int \left( \frac{d\sigma(x)}{dt} \right)^2 dt} Dx. \quad (2.21)$$

If it is assumed that on the phase space no point should be distinguishable from any other point then the metric  $d\sigma(x)^2$  on the phase space should be homogeneous. Therefore, the metric should be chosen such that the resulting geometry has constant curvature. Different choices of the geometry lead to different kinematical variables on which we quantize the system, for further details see [9]. For example, the flat case leads to quantization with the ordinary Heisenberg pair of operators. The constant positive curvature case leads to an underlying quantum kinematical spin operators  $S_i$  where  $[S_i, S_j] = i\varepsilon_{ijk}S_k$ .

The measure in (2.9) can now be replaced by the well defined pinned Wiener measure (2.21). This is done by the addition of the extra factor

$$e^{-\frac{1}{2\nu} \int \left(\frac{d\sigma}{dt}\right)^2 dt}. \quad (2.22)$$

In the limit  $\nu \rightarrow \infty$ , this term vanishes and we are left with our original path integral. In this limit, Daubechies and Klauder [8] showed that the appropriate symbol to use is the lower symbol  $h(x)$ .

$$\langle x'' | e^{-\frac{i}{\hbar} \hat{H}T} | x' \rangle = \lim_{\nu \rightarrow \infty} \mathcal{N} \int d\mu_W^\nu(x) \exp -\frac{i}{\hbar} \left\{ \int i\hbar \langle x | \frac{d}{dt} | x \rangle + h(x) \right\} \quad (2.23)$$

I would now like to consider this formalism with a system of first class constraints.

### 3 Classical Constraints

To begin with, let us consider a  $2M$  dimensional phase space labeled by coordinates  $y^a$ . On this phase space, we will consider a system of  $N$  first class constraints given by  $\phi_i(y^a) = 0$ . Let these constraints form a closed algebra with respect to the Poisson Bracket,

$$[\phi_i, \phi_j] = C_{ij}^k \phi_k. \quad (3.1)$$

Let them also be complete. In other words, the constraints also commute with the total Hamiltonian on the constraint surface such that the time evolution does not generate further constraints.

$$[\phi_i, H] = \frac{d\phi_i}{dt} \approx 0 \quad (3.2)$$

Such constraints can always be Abelianized locally by a canonical transformation [10]. However, the local coordinate patch may not cover the entire constraint surface. For this paper we will assume that we can work on one coordinate patch. After Abelianization, the constraint equation can now be written as

$$p_i = 0, \quad i = 1, \dots, N. \quad (3.3)$$

The gauge orbits are then along  $q^i$ . The reduced phase space can be labeled by  $(2M - 2N)$  variables  $z^b$ . The Poisson bracket algebra becomes

$$[p_i, q^j] = \delta_i^j \quad [q^i, z^a] = 0 \quad [p_i, z^a] = 0 \quad [z^a, z^b] = C_c^{ab} z^c. \quad (3.4)$$

The normal coordinates  $(p_i, q^j)$  close under the Poisson bracket and commute with the reduced phase space variables  $(z^a)$ . We will now map this set of coordinates onto their related operator such that

$$[\hat{Q}^i, \hat{P}_j] = i\hbar\delta_j^i \quad [\hat{Q}^i, \hat{Z}^a] = 0 \quad [\hat{P}_i, \hat{Z}^b] = 0. \quad (3.5)$$

The operators  $\hat{Z}^a$  can also be broken into pairs of Heisenberg operators. However, we will not need to make use of this other than the fact that the reduced phase space has its own coherent state representation.

$$|z^a\rangle \equiv U(z^a)|\eta\rangle \quad (3.6)$$

We will make use of this reduced phase space coherent state throughout out the paper.

The coherent state on the full phase space may be written as

$$|y\rangle \equiv |p_i, q^j, z^b\rangle = e^{if(p, q, z)} e^{-\frac{i}{\hbar} q^j \hat{P}_j} e^{\frac{i}{\hbar} p_i \hat{Q}^i} U(z^b) |\eta', \eta\rangle. \quad (3.7)$$

where  $|\eta', \eta\rangle = |\eta'\rangle \otimes |\eta\rangle$ , the direct product of the fiducial vector for the reduced phase space and the a fiducial vector for the normal coordinates  $(p_i, q^j)$ . The first term is just a phase factor. It appears in the path integral in terms of the symplectic one form which determines which coordinate will play the roll of the momentum and position. This one form can be changed by the addition of a total derivative in the action (see [3]). We will assume that we are free do this. The result being that we can write the above coherent state as the direct product of the reduced phase space coherent state with the normal coordinates coherent state.

$$\begin{aligned} |p_i, q^j, z^b\rangle &= |p_i, q^j\rangle \otimes |z^a\rangle \\ |p_i, q^j\rangle &= e^{-\frac{i}{\hbar} q^j \hat{P}_j} e^{\frac{i}{\hbar} p_i \hat{Q}^i} |\eta'\rangle \end{aligned} \quad (3.8)$$

So we have constructed a coherent state in the new coordinate system. We would now like to show that this coherent state admits a resolution of unity. In the original coordinate system  $(y'^b)$  before we Abelianized the constraints, the coherent state did admit a resolution of unity.

$$(p'_i, q'^j) = (y'^a) \quad i, j = 1, \dots M \quad (3.9)$$

$$|p'_i, q'^j\rangle = e^{-\frac{i}{\hbar} q'^j \hat{P}'_j} e^{\frac{i}{\hbar} p'_k \hat{Q}'^k} |\eta\rangle \quad (3.10)$$

$$\mathbb{I} = \int |p'_i, q'^j\rangle \langle p'_i, q'^j| \prod_{i,j=1}^M \frac{dp'_i dq'^j}{2\pi} \quad (3.11)$$

The new measure  $d\mu(q, p) = d\mu(q', p')$  because the transformation is canonical and therefore the Jacobian is one. The resolution of unity on the new set of coordinates is

$$\mathbb{I} = \int |p_i, q^j, z^a\rangle \langle p_i, q^j, z^a| \left( \prod_{i,j=1}^N \frac{dp_i dq^j}{2\pi} \right) d\mu(z^a) \quad i, j = 1, \dots, N. \quad (3.12)$$

We must take care because the transformation may not be one to one. The above integration is on the whole of the phase space and not on the image of the original phase space. Now we would like to apply the constraint to this coherent state.

Because the constraint equation (3.3) is originally a classical object, there are two ways in which we may apply the constraint. The first is to apply the constraints to the classical variables  $p_i$ . We will consider this case in this section. The other way is demand that the operators  $\hat{P}_i$  on physical states is zero, similar to Dirac quantization. We will consider this approach in the next section.

The classically constrained coherent state ( $p_i = 0$ ) becomes

$$|q^i, z^b\rangle \equiv |p_i = 0, q^j, z^b\rangle = e^{-\frac{i}{\hbar} q^j \hat{P}_j} |\eta'\rangle \otimes |z^a\rangle. \quad (3.13)$$

We want to choose a fiducial vector such that  $\langle \eta | \hat{P}_i | \eta \rangle = \langle \eta | \hat{Q}_i | \eta \rangle = 0 \ \forall i$ . Such a vector is called physically centered, and can always be found [1]. Now, looking at the expectation values of  $\hat{P}_i$ , we see that the constraint becomes fuzzy (higher orders terms of the constraint operators are not zero).

$$\begin{aligned} \langle q^i, z^a | \hat{P}_i | q^i, z^a \rangle &= \langle \eta | \hat{P}_i | \eta \rangle = 0 \\ \langle q^i, z^a | \hat{P}_i^2 | q^i, z^a \rangle &= \langle \eta | \hat{P}_i^2 | \eta \rangle = \frac{\hbar}{2} \\ &\vdots \\ \langle q^i, z^a | \hat{P}_i^n | q^i, z^a \rangle &= \langle \eta | \hat{P}_i^n | \eta \rangle = \mathcal{O}(\hbar) \end{aligned} \quad (3.14)$$

The classical constraints can be understood as fixing the center of a wave packet instead of forcing the wave function to collapse into an eigenstate of the constraint operators.

We have seen how to construct states that classically satisfy the first class constraints. Next, let us consider how we can construct the path integral using these states. We can use the resolution of unity on the full phase space to construct the



path integral. Then at each time step in (2.6) or (2.13), we will force the constraint equation to be obeyed by projecting onto  $p_i = 0$  with the standard delta function. Because we are only dealing with first class constraints, we do not have to include another term such as a determinate for second class constraints in our path integral.

$$\delta(p_i) = \int_{-\infty}^{\infty} d\lambda^i e^{-i\lambda^i p_i}. \quad (3.15)$$

$$\langle y'' | e^{i\hat{H}(t''-t')} | y' \rangle = \int \dots \int \prod_{n=1}^N d\mu(y_n) d\lambda_n^i \prod_{n=0}^N \langle y_{n+1} | e^{-\frac{i\varepsilon}{\hbar} \hat{H}} e^{-i\lambda_n^i p_{i_n}} | y_n \rangle \quad (3.16)$$

Now we rescale  $\lambda$  by  $\hbar/\varepsilon$  and adjust the measure accordingly.

$$\begin{aligned} \langle y'' | e^{i\hat{H}(t''-t')} | y' \rangle &= \int \dots \int \prod_{n=1}^N d\mu(y_n) d\lambda_n^i \prod_{n=0}^N \langle y_{n+1} | e^{-\frac{i\varepsilon}{\hbar} (\hat{H} + \lambda_n^i p_{i_n})} | y_n \rangle \\ y_0 = y' = (p = 0, q', z') \quad y_{N+1} = y'' = (p = 0, q'', z'') \end{aligned} \quad (3.17)$$

In the continuum limit, the path integral becomes

$$\begin{aligned} \int Dy D\lambda \exp -\frac{i}{\hbar} \left[ \int i\hbar \langle y | \frac{d}{dt} | y \rangle + H_T(y) \right] \\ \text{where} \quad H_T(y) = H(y) + \lambda^i p_i. \end{aligned} \quad (3.18)$$

With a physically centered fiducial vector, the symbol  $H_T(y)$  can be defined as  $\langle y | \hat{H}(y) + \lambda^i \hat{P}_i | y \rangle$  which is equal to symbol above (3.18). Because the Lagrange multiplier term is linear in  $p_i$ , this term is equivalent when we switch between upper and lower symbols. So the lower symbol for the total Hamiltonian is  $h(y) + \lambda(y)^i p_i(y)$ .

At this point, we have derived the path integral for the total Hamiltonian. If we apply a stationary phase approximation, we get the normal classical equation of motion plus the constraint equations  $p_i = 0$ . However, we would like to change to the pinned Wiener measure in the above path integral. In addition we would like to include the extended variables  $\lambda^i$  into this measure. We would also like to find the reduced Hamiltonian and its associated path integral. In so doing, we must take care to choose a suitable metric for the Wiener measure.

To find the reduced phase space Hamiltonian and the associated path integral, let us return to the lattice path integral (3.17). Before taking the continuum limit, we want the states at each time slice to satisfy the constraint equation ( $p_i = 0$ ). Integrate along  $\lambda_n^i$  leads to a  $\delta(p_{i_n})$  at time step.

$$\langle y'' | e^{-\frac{i}{\hbar} \hat{H}(t''-t')} | y' \rangle = \int \dots \int \prod_{n=1}^N dy_n \delta(p_{i_n}) \prod_{n=0}^N \langle y_{n+1} | e^{-\frac{i\varepsilon}{\hbar} \hat{H}} | y_n \rangle \quad (3.19)$$

Now, because we have removed the extended variables, we can take the continuum limit and change the measure to the Wiener measure.

$$\langle y'' | e^{-\frac{i}{\hbar} \hat{H}(t''-t')} | y' \rangle = \int d\mu_W^\nu(y) \delta[p_i] \exp -\frac{i}{\hbar} \left\{ \int i\hbar \langle y | \frac{d}{dt} | y \rangle + h(y) \right\} \quad (3.20)$$

If we had just transformed the labeling space of the coherent state by defining  $|\tilde{y}\rangle \equiv |y\rangle$  as in [5]. Then we would have to carry over the metric for the Wiener measure (2.21) from the original phase space to the new coordinates. Klauder calls this a shadow metric. However we have switched our coherent state to be defined on the new phase space, and we have replaced the operators with the new operators (3.5). Because of this, it is not clear which metric should be placed on this new phase space. The metric should however be compatible with our constraints.

We wish to integrate the path integral in the  $p_i$  direction to remove the delta function and fix  $p_i = 0$ . Because we do not want to introduce any extraneous coupling between the reduced phase space coordinates ( $z$ ) and the normal coordinates ( $p, q$ ) let us choose a metric on the phase space that can be separated,

$$d\sigma(y)^2 = d\sigma(z)^2 + d\sigma(p, q)^2. \quad (3.21)$$

In addition, we want the metric  $d\sigma(p, q)^2$  to be consistent with the gauge transforms ( $q^i \rightarrow q^i + f^i$ ). Therefore, the metric should be independent of  $q^i$ . Also, the metric should be well defined for  $p_i = 0$ . For example, in two dimensions, we can choose the metric to have the form

$$d\sigma(p, q)^2 = g_{11}(p)dp^2 + 2g_{12}(p)dpdq + g_{22}(p)dq^2. \quad (3.22)$$

Now, we can integrate over the  $p$ . The delta function will fix  $p = 0$  for each time slice. So the metric will become

$$d\sigma(q)^2 = g_{22}dq^2, \quad (3.23)$$

where  $g_{22}$  is now just a constant.

Now, we can also integrate along the gauge orbits  $q^i$ . Because the integral is regularized, on first appearances, we do not have to gauge fix these orbits to remove the infinity redundancies. Because  $g_{22}$  is just a constant on the constraint surface, we can rescale  $\nu$  and/or  $q^i$ , such that the metric is just  $dq^2$ . The measure for our Wiener measure should then take the form

$$d\mu_W^\nu(z, q) = \mathcal{N} e^{-\frac{1}{2\nu} \int \left( \frac{d\sigma(z)}{dt} \right)^2 + \left( \frac{dq}{dt} \right)^2} Dz Dq. \quad (3.24)$$

Turning to the terms in the exponent in the path integral, the term  $\langle y | \frac{d}{dt} | y \rangle$  can be written in the following way using the definition (3.8).

$$\langle y | \frac{d}{dt} | y \rangle = -\frac{i}{\hbar} p_j \frac{dq^j}{dt} + \langle z | \frac{d}{dt} | z \rangle \quad (3.25)$$

With  $p_i = 0$  the first term drops out, and we can write the path integral in the continuum limit as

$$\int d\mu_W^\nu(z, q) \exp \left\{ -\frac{i}{\hbar} \int \left( i\hbar \langle z | \frac{d}{dt} | z \rangle + h(z, q) \right) dt \right\}. \quad (3.26)$$

The symbol  $h(q, z)$  is defined from the full phase space symbol  $h(p, q, z)$  with  $p_i$  set equal to zero. All though there is no  $p_i$  dependence in the symbol, there may still be  $q^i$  dependence. We will discuss this point in a bit. For now, let us try to extend the Wiener measure to the extended phase space.

Let us go back to our earlier definition of the path integral (3.18), and instead of fixing the states at each time slice, we will work on the extended phase space and its associated total Hamiltonian. In so doing, we should regularize the new measure which includes the extended coordinates. Because of the of the above construction, it is natural to choose the metric on this extended phase space as

$$d\sigma(y, \lambda)^2 = d\sigma(z)^2 + d\sigma(p, q)^2 + d\lambda^2. \quad (3.27)$$

The Wiener measure now takes the form

$$d\mu_W^\nu(y, \lambda) = \mathcal{N} e^{-\frac{1}{2\nu} \int \left( \left( \frac{d\sigma(z)}{dt} \right)^2 + \left( \frac{d\sigma(p, q)}{dt} \right)^2 + \left( \frac{d\lambda}{dt} \right)^2 \right) dt} Dz Dp Dq D\lambda, \quad (3.28)$$

and the path integral (3.17) becomes

$$\int d\mu_W^\nu(y, \lambda) \exp \left\{ -\frac{i}{\hbar} \int \left( i\hbar \langle y | \frac{d}{dt} | y \rangle + h(y) + \lambda^i p_i \right) dt \right\}. \quad (3.29)$$

We can now integrate  $\lambda$  with our definition above. Let us work out an example with only one constraint and a flat metric on the constraint variables  $d\sigma(p, q)^2 = dp^2 + dq^2$ . The only terms involving  $\lambda$  are

$$\begin{aligned} \int d\mu_W^\nu(\lambda) \exp(i\lambda p) &= \mathcal{N} \int \prod_{n=1}^N d\lambda_n \left( \frac{1}{2\pi\nu\varepsilon} \right)^{\frac{N}{2}} \\ &\exp \left( \sum_{n=0}^N -\frac{(\lambda_{n+1} - \lambda_n)^2}{2\nu\varepsilon} - \frac{i}{\hbar} \lambda_n p_n \right). \end{aligned} \quad (3.30)$$

Integrating over one of the  $\lambda_n$ 's,

$$\begin{aligned}
& \int_{-\infty}^{\infty} d\lambda_n \left( \frac{1}{2\pi\nu\varepsilon} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2\nu\varepsilon} \left( (\lambda_{n+1} - \lambda_n)^2 + (\lambda_n - \lambda_{n-1})^2 \right) - \frac{i}{\hbar} \lambda_n p_n \right] \\
&= \frac{1}{\sqrt{2}} \exp \left[ -\frac{1}{2\nu\varepsilon} \left( \frac{1}{2} (\lambda_{n+1} - \lambda_{n-1})^2 + \frac{i}{\hbar} \varepsilon \nu p_n (\lambda_{n+1} + \lambda_{n-1}) + \frac{\varepsilon^2 \nu^2}{2\hbar^2} p_n^2 \right) \right]. \quad (3.31)
\end{aligned}$$

If we were to take the limit  $\nu \rightarrow \infty$  at this stage, the last term ( $p_n^2$ ) dominates, and we see that the delta function is still hidden in this expression.

$$\lim_{\nu \rightarrow \infty} \exp -\nu p_n^2 = \sqrt{\frac{\pi}{\nu}} \delta(p_n). \quad (3.32)$$

However, we should consider taking the limit of all the  $\mu$  together. So let us continue to integrate over all possible  $\lambda_n$ 's without taking the limit yet. In so doing

$$\begin{aligned}
\int d\mu_W^\nu(\lambda) \exp(-\frac{i}{\hbar} \lambda p) &= \mathcal{N} \left( \frac{1}{2\pi(t_{N+1} - t_0)} \right)^{\frac{1}{2}} \exp \left[ -\frac{(\lambda_{N+1} - \lambda_0)^2}{2\nu(t_{N+1} - t_0)} \right] \\
&\exp \left[ \frac{1}{(t_{N+1} - t_0)} \left[ \left( \sum_{j=1}^N -\frac{i}{\hbar} p_j(t_j - t_0) \right) \lambda_{N+1} + \left( \sum_{j=0}^{N-1} -\frac{i}{\hbar} p_j(t_{N+1} - t_j) \right) \lambda_0 \right] \right] \\
&\exp \left[ -\frac{\nu}{\hbar^2(t_{N+1} - t_0)} \sum_{j,k=1}^{N-1} p_j p_k (t_{N+1} - t_j)(t_k - t_0) \right]. \quad (3.33)
\end{aligned}$$

In the continuum limit, this becomes

$$\begin{aligned}
\int d\mu_W^\nu(\lambda) \exp(-\frac{i}{\hbar} \lambda p) &= \mathcal{N} \left( \frac{1}{2\pi(t_{N+1} - t_0)} \right)^{\frac{1}{2}} \exp \left[ -\frac{(\lambda_{N+1} - \lambda_0)^2}{2\nu(t_{N+1} - t_0)} \right] \\
&\exp \left[ \frac{1}{(t_{N+1} - t_0)} \left( \int -\frac{i}{\hbar} p(t)(t - t_0) dt \right) \lambda_{N+1} + \left( \int -\frac{i}{\hbar} p(t)(t_{N+1} - t) dt \right) \lambda_0 \right] \\
&\exp \left[ -\frac{\nu}{\hbar^2(t_{N+1} - t_0)} \int_{t_0}^{t_{N+1}} \int_{t_0}^t p(t)p(s)(t_{N+1} - t)(s - t_0) ds dt \right]. \quad (3.34)
\end{aligned}$$

Because  $\lambda$  is just a Lagrange multiplier, after integrating, the final answer should not be dependent on its initial or final value. The first term in the limit  $\nu \rightarrow \infty$  is zero if  $\lambda_N + 1 \neq \lambda_0$ . This forces  $\lambda_N + 1 = \lambda_0$ . This would seem to make it impossible to fold together two propagators to make a propagator over a longer time.

$$\langle y''' | e^{-\frac{i}{\hbar} \hat{H}(t''' - t')} | y' \rangle = \int Dy'' D\lambda \langle y''' | e^{-\frac{i}{\hbar} \hat{H}(t''' - t'')} | y'' \rangle \langle y'' | e^{-\frac{i}{\hbar} \hat{H}(t'' - t')} | y' \rangle \quad (3.35)$$

But this condition is forced on us at the last step of the integration and does not need to be set before hand. So we can fold together two propagators and as a last step let  $\lambda_{N+1} = \lambda_0$ .

The second term in (3.34) gives us a possible phase factor.

$$\exp \left[ i\lambda \int p(t)dt \right] = \exp \left[ i\lambda p_{\text{ave}}(t'' - t') \right] \quad (3.36)$$

The last term can be expanded in terms of a saddle point approximation as  $\nu$  becomes large.

$$S[p] = \int \int \left( p(t)p(s)\theta(s-t)(t_{N+1}-t)(s-t_0) \right) ds dt \quad (3.37)$$

$$\Rightarrow \quad \frac{\delta S[p]}{\delta p} = p(s) \int \left( \theta(s-t)(t_b-t)(s-t_a) + \theta(t-s)(t_b-s)(t-t_b) \right) dt = 0 \quad (3.38)$$

This integrand is always positive. Therefore, for  $\delta p$  arbitrary, this implies that  $p(t) = 0$ . Once again, we get back to our original delta function. With this delta function, as we integrate our path integral along  $p$ , we see that the second term (3.36) also drops out. The extended phase space, with the above choice for the metric on the phase space leads to an equivalent path integral to the one derived before (3.26).

Either integrating over each time slice or choosing to extend our phase space, the resulting “classical” reduced Hamiltonian symbol  $H_{cr}$  takes the form

$$\begin{aligned} H_{cr} &\equiv \langle p, q, z | H(\hat{P}_i, \hat{Q}^i, \hat{Z}^a) | p, q, z \rangle \Big|_{p_i=0} \\ &= \langle z | \otimes \langle \eta | H(\hat{P}_i, \hat{Q}^i + q^i \mathbb{I}, \hat{Z}^a) | \eta \rangle \otimes | z \rangle. \end{aligned} \quad (3.39)$$

Note, if  $H(\hat{P}_i, \hat{Q}^i, \hat{Z}^a)$  contains terms like  $F(\hat{Q})\hat{P}^2$ , such a term, for example, might occur in the kinematic term on a curved surface, then

$$\begin{aligned} \tilde{H} &= \langle q, z | F(\hat{Q})\hat{P}^2 | q, z \rangle \\ &= \langle \eta | F(\hat{Q} + q\mathbb{I})\hat{P}^2 | \eta \rangle \\ &= \langle \eta | \left( F(q) + \frac{\partial F}{\partial Q} \Big|_{Q=q} \hat{Q} + \dots \right) \hat{P}^2 | \eta \rangle \\ &= F(q) \langle \eta | - \frac{\hbar}{2} (a - a^\dagger)^2 | \eta \rangle + \mathcal{O}(\hbar^2) \\ &= \frac{\hbar}{2} F(q) + \mathcal{O}(\hbar^2). \end{aligned} \quad (3.40)$$

The classically reduced Hamiltonian symbol  $H_{cr}$  is still in general dependent on the  $q^i$  the gauge orbits. However such a term can only appear for terms of order  $\hbar$  or higher.

$$\begin{aligned} H_{cr}(z, q) &= \langle z | H(\hat{Z}^a) | z \rangle + \hbar H_1(z, q) \\ &= H_0(z) + \hbar H_1(z, q), \end{aligned} \quad (3.41)$$

where  $H_0$  is the classical Hamiltonian on the reduced phase space. Even if there is no dependence on  $q^i$  in  $H_1$ , such a term will not, for most reasonable Hamiltonians, be zero. Likewise for the lower symbol.

$$\begin{aligned} \tilde{H}' &= \int p^2 f(q) |p, q\rangle \langle p, q| dp dq \\ &= \hat{P}^2 F(\hat{Q}) + \mathcal{O}(\hbar) \end{aligned} \quad (3.42)$$

If such a term should appear, it would break the original gauge symmetry. Because this gauge breaking term(s) will be of the order  $\hbar$ , it will not appear in the equations of motion. However, the breaking of the gauge symmetry will lead to a classical observable effect. This implies that we have not correctly chosen the Hamiltonian operator for this system. If the classical Hamiltonian is a polynomial in  $p$  and  $q$ , then we can clearly redefine the Hamiltonian operator in terms of the above spectral representation without any problems. In so doing, we are left with only the reduced phase space Hamiltonian  $h_0(z)$ .

At this point, we must still integrate along the gauge orbits in general. So our path integral is

$$\begin{aligned} \mathcal{N} \int d\mu_W^\nu(z, q) \exp \left\{ -\frac{i}{\hbar} \int i\hbar \langle z | \frac{d}{dt} | z \rangle + h_0(z) dt \right\} \\ d\mu_W^\nu(z, q) = \mathcal{N} e^{-\frac{1}{2\nu} \int \left( \frac{d\sigma(z)^2}{dt} \right) + \left( \frac{dq^2}{dt} \right) dt} Dz Dq \end{aligned} \quad (3.43)$$

Now that we have removed all the gauge dependencies. We can integrate along gauge orbits. Because the Wiener measure gives us a finite volume for this integration, we do not have to gauge fix this path integral. Instead, we can just integrate the volume and absorb it into the normalization constant. This leaves us with the reduced phase space path integral

$$\begin{aligned} \langle y'' | e^{-\frac{i}{\hbar} \hat{H}_T(t''-t')} | y' \rangle &= \mathcal{N}' \int d\mu_W^\nu(z) \exp \left\{ -\frac{i}{\hbar} \int i\hbar \langle z | \frac{d}{dt} | z \rangle + h_0(z) dt \right\} \\ &= \langle z'' | e^{-\frac{i}{\hbar} \hat{H}_0(t''-t')} | z' \rangle \end{aligned} \quad (3.44)$$

## 4 Quantum Constraints

Another approach to applying the constraints is to define the physical states as being annihilated by the constraint operator as in Dirac quantization. For our constraint ( $p_i = 0$ ), there is no problem with factor ordering. So we can define the physical states as

$$\hat{P}_i |\psi_{phys}\rangle = 0. \quad (4.1)$$

These constraints are sharp, unlike the earlier classical constraint (3.14). This is because we are now forcing the state to collapse into a momentum eigenstate.

The physical state is therefore the zero momentum eigenstate and the reduced phase space coherent state,

$$|q, p, z\rangle_{phys} = |p_i = 0\rangle \otimes |z^a\rangle. \quad (4.2)$$

Let us see what happens when we try to write this zero momentum eigenstate in terms of the earlier coherent state form (3.8).

$$\begin{aligned} \hat{P}_j |q^i, p_j, z^a\rangle_{phys} &= \hat{P}_j \left( e^{-\frac{i}{\hbar} q^j \hat{P}_j} e^{\frac{i}{\hbar} p_i \hat{Q}^i} |\eta'\rangle \right) \otimes |z^a\rangle \\ &= \left( p_j e^{-\frac{i}{\hbar} q^j \hat{P}_j} e^{\frac{i}{\hbar} p_i \hat{Q}^i} |\eta'\rangle + e^{-\frac{i}{\hbar} q^j \hat{P}_j} e^{\frac{i}{\hbar} p_i \hat{Q}^i} \hat{P}_j |\eta'\rangle \right) \otimes |z^a\rangle \\ &= 0 \\ \Rightarrow \quad \hat{P}_j |\eta'\rangle &= -p_j |\eta'\rangle \end{aligned} \quad (4.3)$$

So the fiducial vector must also be an eigenstate of the momentum operator. Continuing, we can further simplify by changing the order of the terms in the coherent state. Note also that the position operator generates translations in the momentum space.

$$\begin{aligned} |q^i, p_j, z^a\rangle_{phys} &= e^{-\frac{i}{\hbar} q_i p^i} e^{\frac{i}{\hbar} p_j \hat{Q}^j} e^{-\frac{i}{\hbar} q^k \hat{P}_k} | -p_j \rangle \otimes |z^a\rangle \\ &= e^{\frac{i}{\hbar} p_j \hat{Q}^j} | -p_j \rangle \otimes |z^a\rangle \\ &= | -p_j + p_j \rangle = | p = 0 \rangle \end{aligned} \quad (4.4)$$

So we have come full circle and back to the zero momentum eigenstate.

These constraint equations also can be solved easily in the Shrödinger representation. We keep the same ordering of the coherent state as defined in (3.8).

$$\hat{Q}^i = x^i \quad \hat{P}_i = -i\partial/\partial x^i$$

$$\begin{aligned}\langle x^i | q, p \rangle &= \psi(x^i) = e^{-iq^i \hat{P}_i} e^{ip^i \hat{Q}_i} \phi(x^i) = e^{ip_i q^i} e^{ip_i x^i} \phi(x^i - q^i) \\ \int d^N x \overline{\phi(x^i)} \phi(x^i) &= 1\end{aligned}\tag{4.5}$$

The quantum constraint equation (4.1) is then solved by

$$-i \frac{\partial}{\partial x^i} \psi(x^i) = 0 \quad \implies \quad \psi(x^i) = \text{const.}\tag{4.6}$$

Both the  $q^i$  and  $p_i$  dependencies drop out of the coherent state. We are left with a coherent state that is only dependent on the reduced phase space coordinates. However, this reduced phase space coherent is no longer normalizable on the full phase space. To deal with this, we can let  $x$  be restricted to a finite box. Then let the box go to infinity. The normalized physical wave function is

$$\langle x | q, p, z \rangle_{phys} = \frac{1}{\sqrt{Vol}} \langle x | z \rangle\tag{4.7}$$

$$\implies \quad |p, q, z\rangle_{phys} \approx |z\rangle.\tag{4.8}$$

This leads us to using the reduced phase space coherent state and its resolution of unity to construct path integral that only involves the reduced Hamiltonian. However we will see that we can still use the full space time to construct a path integral.

The resolution of unity on the full phase space can be shown to preserve the form of the physical state.

$$|p = 0\rangle = \int |p', q'\rangle \langle p', q' | p = 0\rangle \frac{dp' dq'}{2\pi}\tag{4.9}$$

We can also solve part of the Hamiltonian metric element with out resorting to the using the path integral. Because the physical state is the zero eigenstate of momentum, all terms with  $\hat{P}$  drop out. Because we are assuming only first class constraints this means that all terms with  $\hat{Q}$  also drop out. So we are left with the reduced Hamiltonian.

$$\begin{aligned}\langle p', q', z' | e^{-\frac{i}{\hbar} H(\hat{P}, \hat{Q}, \hat{Z})^T} | p, q, z \rangle_{phys} &= \langle z' | \otimes \langle p = 0 | e^{-\frac{i}{\hbar} H(\hat{P}, \hat{Q}, \hat{Z})^T} | p = 0 \rangle \otimes | z \rangle \\ &= \langle z' | e^{-\frac{i}{\hbar} H_0(\hat{Z})^T} | z \rangle\end{aligned}\tag{4.10}$$

Because the resolution of unity preserves the physical state and the reduced Hamiltonian does not act upon the normal coordinates, we can write the path integral on the full phase space. There are no time dependencies in the normal coordinates so the first term also only depends on the reduced coordinates.



$$\langle y'' | e^{-\frac{i}{\hbar} \hat{H}_T(t''-t')} | y' \rangle_{phys} = \mathcal{N} \int d\mu_W^\nu(p, q, z) \exp \left\{ -\frac{i}{\hbar} \int i\hbar \langle z | \frac{d}{dt} | z \rangle + h_0(z) dt \right\} \quad (4.11)$$

If the assumptions that the Wiener measure does not involve cross terms between  $z$  and  $p$  and  $q$ , the integration over  $(p, q)$  in the above path integral only give us a volume and can be absorbed into the normalization. Once again we arrive at the reduced phase space path integral.

$$\langle y'' | e^{-\frac{i}{\hbar} \hat{H}_T(t''-t')} | y' \rangle = \mathcal{N}' \int d\mu_W^\nu(z) \exp \left\{ -\frac{i}{\hbar} \int i\hbar \langle z | \frac{d}{dt} | z \rangle + h_0(z) dt \right\} \quad (4.12)$$

## 5 Discussion

Either using the classical constraints or the quantum constraints, we are lead to the reduced phase space path integral. The above derivations relied on the fact that we can find a set of coordinates for which the constraint equation become simple ( $p = 0$ ). It is easy to see that the classical constraint approach can easily be carried over to constraints that are not just linear in the momentum but may depend on higher orders of  $p$  and  $q$ . The momentum symbol in the path integral would just be replaced by the symbol for the constraint. Ordinarily it is difficult to repeat the above solution for a quantum constraint that is not just linear in the momentum. In this case, it is not clear which ordering of the operator to use. Also it may not be possible to solve the equation for the physical states. However, Klauder in a recent pre-print [13] has constructed a operator version of the projection operator (3.15). This projection operator ties together these two approaches.

The fact that we arrive at the reduced phase space path integral should not be surprising with the assumptions we have made. These assumptions lead to a simple topology on the phase space. The difference between Dirac quantization and reduced phase space tends to appear only with non-trivial topology (see for examples [14]). It should be possible to extend this ideal to work on a set of coordinate patches, then we can use constraint coherent states to study some examples where the reduced phase space and Dirac quantization do not agree. In these cases a difference between the classical constraints and the quantum constraints may appear. If these two approaches do not agree then the coherent states would give a direct relationship between Dirac and reduced phase quantization.

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